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2006 J. Phys.: Condens. Matter 18 10631

(http://iopscience.iop.org/0953-8984/18/47/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **18** (2006) 10631–10641 doi:10.1088/0953-8984/18/47/009

Stress potential function formulation and complex variable function method for solving the elasticity of quasicrystals of point group 10 and the exact solution for the notch problem

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Received 10 May 2006, in final form 20 October 2006
Published 13 November 2006
Online at stacks.iop.org/JPhysCM/18/10631

Abstract
The plane elasticity equations of two-dimensional quasicrystals of point group 10 are reduced to a single partial differential equation with eighth order by introducing a stress potential function. Further, we develop the complex variable function method for classical elasticity theory to that of the quasicrystals. The complex representations of stress and displacement components of phonon and phason fields in the quasicrystals are given. With the help of conformal transformation, an exact solution for the elliptic notch of the quasicrystals is presented. The solution of the Griffith crack problem as a special case of the results is also observed. This work shows that the stress potential and complex variable function methods are powerful for solving the complicated boundary value problems of higher order partial differential equations originating from quasicrystal elasticity.

1. Introduction

Quasicrystals were first observed as a new structure by Shechtman *et al* [1] and announced in 1984. Since their discovery, they have attracted the extensive attention of researchers in both experimental and theoretical work. The mechanical behaviour of the new solid phase, in which elasticity and defects play a central role, is of fundamental importance. The elasticity theory was discussed in many references (e.g. [2–9]), but some investigators developed various methods ranging from the iterative method [10] and the Green function method [11] to the Fourier transform method [12–14] for solving the relevant elasticity problems. Using these methods, considerable exact analytic solutions for some dislocation and crack problems of two-dimensional quasicrystals have been constructed. In [12–14] a so-called displacement potential function was used and greatly simplified the complicated equations involving the elasticity. In quite a different version of these studies, Fan [15] developed a complex variable function method with conformal mapping for solving the elasticity and crack problems of one-dimensional hexagonal quasicrystals for static and dynamic cases. Afterwards, Liu and Fan [16, 17] further developed the method for plane elasticity of other one-dimensional quasicrystals and two-dimensional quasicrystals of point group $10mm$. In the review given by Fan and Mai [18] some summarization for the above development are addressed.

The elasticity of quasicrystals of point group 10 is more complicated than that of one-dimensional quasicrystals and two-dimensional quasicrystals of point group $10mm$. Methods listed in the above references could not solve the notch problem for these kinds of materials. For this reason, we here first introduce a stress potential function and obtain a simple and effective formulation for the present problem. Furthermore, we develop a new complex function method for treating more general complicated boundary value problems. A series of exact solutions for notch and crack problems are constructed. The computational results show that the formulation and the method are powerful for solving some higher order partial differential equations coupled with complicated boundary conditions, which originate from the elasticity and defect problems of quasicrystals.

2. General solution

Assume that the atom arrangement along the z-direction is periodic and along the x–y plane is quasiperiodic, and denote $x = x_1$, $y = x_2$ and $z = x_3$ for a two-dimensional decagonal quasicrystal with point group 10. If the x_3-axis represents the 10-fold symmetry axis, according to quasicrystal elasticity theory [9] we have the equations of deformation geometry

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad w_{ij} = \frac{\partial w_i}{\partial x_j} \tag{1}$$

the equilibrium equations (if the body force is neglected)

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad \frac{\partial H_{ij}}{\partial x_j} = 0 \tag{2}$$

and the generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl}, \qquad H_{ij} = R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl} \tag{3}$$

where u_i and w_i denote the phonon and phason displacements, σ_{ij} and ε_{ij} the phonon stresses and strains, H_{ij} and w_{ij} phason stresses and strains and C_{ijkl}, K_{ijkl} and R_{ijkl} the phonon, phason and phonon–phason coupling elastic constants respectively. Assume that a plane notch penetrates through the solid along the period direction (x_3 direction). In this case, it is evident that all the field variables are independent of x_3. Considering the situation, the generalized Hooke's law (3) can be rewritten as follows [15, 18]

$$\sigma_{xx} = L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{xx} + R_1(w_{xx} + w_{yy}) + R_2(w_{xy} - w_{yx}) \tag{4a}$$

$$\sigma_{yy} = L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{yy} - R_1(w_{xx} + w_{yy}) - R_2(w_{xy} - w_{yx}) \tag{4b}$$

$$\sigma_{xy} = \sigma_{yx} = 2M\varepsilon_{xy} + R_1(w_{yx} - w_{xy}) + R_2(w_{xx} + w_{yy}) \tag{4c}$$

$$H_{xx} = K_1 w_{xx} + K_2 w_{yy} + R_1(\varepsilon_{xx} - \varepsilon_{yy}) + 2R_2\varepsilon_{xy} \tag{4d}$$

$$H_{yy} = K_1 w_{yy} + K_2 w_{xx} + R_1(\varepsilon_{xx} - \varepsilon_{yy}) + 2R_2\varepsilon_{xy} \tag{4e}$$

$$H_{xy} = K_1 w_{xy} - K_2 w_{yx} - 2R_1\varepsilon_{xy} + R_2(\varepsilon_{xx} - \varepsilon_{yy}) \tag{4f}$$

$$H_{yx} = K_1 w_{yx} - K_2 w_{xy} + 2R_1\varepsilon_{xy} - R_2(\varepsilon_{xx} - \varepsilon_{yy}) \tag{4g}$$

in which

$$L = C_{12}, \qquad M = (C_{11} - C_{12})/2 = C_{66}. \tag{4h}$$

By the equations of deformation geometry (1), deformation compatibility equations are as follows:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \qquad \frac{\partial w_{xy}}{\partial x} = \frac{\partial w_{xx}}{\partial y}, \qquad \frac{\partial w_{yx}}{\partial y} = \frac{\partial w_{yy}}{\partial x}. \tag{5}$$

If we introduce the stress potential functions $\phi(x, y)$, $\psi_1(x, y)$ and $\psi_2(x, y)$ such as

$$\begin{aligned} &\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \qquad \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y} \\ &H_{xx} = \frac{\partial \psi_1}{\partial y}, \qquad H_{xy} = -\frac{\partial \psi_1}{\partial x}, \qquad H_{yx} = -\frac{\partial \psi_2}{\partial y}, \qquad H_{yy} = \frac{\partial \psi_2}{\partial x} \end{aligned} \tag{6}$$

then equilibrium equations (2) will be automatically satisfied.

Based on the generalized Hooke's law (4) all strain components can be expressed by the relevant stress components:

$$\begin{aligned} \varepsilon_{xx} = \frac{1}{4(L+M)}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{4c}[&(K_1 + K_2)(\sigma_{xx} - \sigma_{yy}) - 2R_1(H_{xx} + H_{yy}) \\ &- 2R_2(H_{xy} - H_{yx})] \end{aligned} \tag{7a}$$

$$\begin{aligned} \varepsilon_{yy} = \frac{1}{4(L+M)}(\sigma_{xx} + \sigma_{yy}) - \frac{1}{4c}[&(K_1 + K_2)(\sigma_{xx} - \sigma_{yy}) - 2R_1(H_{xx} + H_{yy}) \\ &- 2R_2(H_{xy} - H_{yx})] \end{aligned} \tag{7b}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2c}[(K_1 + K_2)\sigma_{xy} - R_2(H_{xx} + H_{yy}) + R_1(H_{xy} - H_{yx})] \tag{7c}$$

$$w_{xx} = \frac{1}{2(K_1 - K_2)}(H_{xx} - H_{yy}) + \frac{1}{2c}[M(H_{xx} + H_{yy}) - R_1(\sigma_{xx} - \sigma_{yy}) - 2R_2\sigma_{xy}] \tag{7d}$$

$$w_{yy} = -\frac{1}{2(K_1 - K_2)}(H_{xx} - H_{yy}) + \frac{1}{2c}[M(H_{xx} + H_{yy}) - R_1(\sigma_{xx} - \sigma_{yy}) - 2R_2\sigma_{xy}] \tag{7e}$$

$$w_{xy} = \frac{1}{2c}[-R_2(\sigma_{xx} - \sigma_{yy}) + 2R_1\sigma_{xy}] + \frac{1}{2(K_1 - K_2)}(H_{xy} + H_{yx}) + \frac{M}{2c}(H_{xy} - H_{yx}) \tag{7f}$$

$$w_{yx} = \frac{1}{2c}[R_2(\sigma_{xx} - \sigma_{yy}) - 2R_1\sigma_{xy}] + \frac{1}{2(K_1 - K_2)}(H_{xy} + H_{yx}) - \frac{M}{2c}(H_{xy} - H_{yx}) \tag{7g}$$

in which

$$c = M(K_1 + K_2) - 2(R_1^2 + R_2^2). \tag{7h}$$

So the deformation compatibility equations (5) can be rewritten by the stresses σ_{ij}, H_{ij}; then by employing (6), one has

$$\begin{aligned} &\left(\frac{1}{2(L+M)} + \frac{K_1 + K_2}{2c}\right)\nabla^2\nabla^2\phi + \frac{R_1}{c}\left(\frac{\partial}{\partial y}\Pi_1\psi_1 - \frac{\partial}{\partial x}\Pi_2\psi_2\right) \\ &\qquad + \frac{R_2}{c}\left(\frac{\partial}{\partial x}\Pi_2\psi_1 + \frac{\partial}{\partial y}\Pi_1\psi_2\right) = 0 \\ &\left(\frac{c}{K_1 - K_2} + M\right)\nabla^2\psi_1 + R_1\frac{\partial}{\partial y}\Pi_1\phi + R_2\frac{\partial}{\partial x}\Pi_2\phi = 0 \\ &\left(\frac{c}{K_1 - K_2} + M\right)\nabla^2\psi_2 - R_1\frac{\partial}{\partial x}\Pi_2\phi + R_2\frac{\partial}{\partial y}\Pi_1\phi = 0 \end{aligned} \tag{8a}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Pi_1 = 3\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \qquad \Pi_2 = 3\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}. \tag{8b}$$

The equations (8) will be satisfied when we choose a new function G, which is called the stress function, such that

$$\begin{aligned}\phi &= c_1\nabla^2\nabla^2 G,\\ \psi_1 &= -\left(R_1\frac{\partial}{\partial y}\Pi_1 + R_2\frac{\partial}{\partial x}\Pi_2\right)\nabla^2 G,\\ \psi_2 &= \left(R_1\frac{\partial}{\partial x}\Pi_2 - R_2\frac{\partial}{\partial y}\Pi_1\right)\nabla^2 G\end{aligned} \tag{9a}$$

in which

$$c_1 = \frac{c}{K_1 - K_2} + M \tag{9b}$$

and

$$\nabla^2\nabla^2\nabla^2\nabla^2 G = 0. \tag{10}$$

The general solution of equation (10) is

$$G = 2\,\mathrm{Re}[g_1(z) + \bar{z}g_2(z) + \tfrac{1}{2}\bar{z}^2 g_3(z) + \tfrac{1}{6}\bar{z}^3 g_4(z)] \tag{11}$$

where $g_i(z)$, $(i = 1, 2, 3, 4)$ are four analytic functions of a single complex variable $z \equiv x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$. The bar denotes the complex conjugate hereinafter, i.e. $\bar{z} = x - \mathrm{i}y = r\mathrm{e}^{-\mathrm{i}\theta}$.

3. The complex representation of stresses and displacements

Substituting expression (11) into (9) then into equation (6) leads to

$$\sigma_{xx} = -32c_1\,\mathrm{Re}(\Omega(z) - 2g_4'''(z)) \tag{12a}$$

$$\sigma_{yy} = 32c_1\,\mathrm{Re}(\Omega(z) + 2g_4'''(z)) \tag{12b}$$

$$\sigma_{xy} = \sigma_{yx} = 32c_1\,\mathrm{Im}\,\Omega(z) \tag{12c}$$

$$H_{xx} = 32R_1\,\mathrm{Re}(\Theta'(z) - \Omega(z)) - 32R_2\,\mathrm{Im}(\Theta'(z) - \Omega(z)) \tag{12d}$$

$$H_{xy} = -32R_1\,\mathrm{Im}(\Theta'(z) + \Omega(z)) - 32R_2\,\mathrm{Re}(\Theta'(z) + \Omega(z)) \tag{12e}$$

$$H_{yx} = -32R_1\,\mathrm{Im}(\Theta'(z) - \Omega(z)) - 32R_2\,\mathrm{Re}(\Theta'(z) - \Omega(z)) \tag{12f}$$

$$H_{yy} = -32R_1\,\mathrm{Re}(\Theta'(z) + \Omega(z)) + 32R_2\,\mathrm{Im}(\Theta'(z) + \Omega(z)) \tag{12g}$$

where

$$\begin{aligned}\Theta(z) &= g_2^{(\mathrm{IV})}(z) + \bar{z}g_3^{(\mathrm{IV})}(z) + \tfrac{1}{2}\bar{z}^2 g_3^{(\mathrm{IV})}(z)\\ \Omega(z) &= g_3^{(\mathrm{IV})}(z) + \bar{z}g_4^{(\mathrm{IV})}(z)\end{aligned} \tag{12h}$$

in which the prime, double prime, triple prime and superscript (IV) denote the first to fourth order differentiation of $g_i(z)$ to variable z; in addition $\Theta'(z) = \mathrm{d}\Theta(z)/\mathrm{d}z$.

We further derive the complex representations of displacement components of phonon and phason fields. Equations (7a) and (7b) can be rewritten as follows

$$\varepsilon_{xx} = c_2(\sigma_{xx} + \sigma_{yy}) - \frac{K_1 + K_2}{2c}\sigma_{yy} - \frac{1}{2c}[R_1(H_{xx} + H_{yy}) + R_2(H_{xy} - H_{yx})] \tag{7a-1}$$

$$\varepsilon_{yy} = c_2(\sigma_{xx} + \sigma_{yy}) - \frac{K_1 + K_2}{2c}\sigma_{xx} + \frac{1}{2c}[R_1(H_{xx} + H_{yy}) + R_2(H_{xy} - H_{yx})] \tag{7b-1}$$

where

$$c_2 = \frac{c + (L + M)(K_1 + K_2)}{4(L + M)c}. \tag{13}$$

Substituting equations (12*a*), (12*b*) and (12*d*)–(12*g*) into (7*a*-1) and applying the first and second formulae of equation (6), we have

$$u_x = 128c_1c_2\,\mathrm{Re}\,g_4''(z) - \frac{K_1+K_2}{2c}\frac{\partial}{\partial x}\phi + \frac{32(R_1^2+R_2^2)}{c}\mathrm{Re}[g_3'''(z) + \bar{z}g_4'''(z) - g_4''(z)] + f_1(y) \tag{7a-2}$$

$$u_y = 128c_1c_2\,\mathrm{Im}\,g_4''(z) - \frac{K_1+K_2}{2c}\frac{\partial}{\partial y}\phi - \frac{32(R_1^2+R_2^2)}{c}\mathrm{Im}[g_3'''(z) + \bar{z}g_4'''(z) + g_4''(z)] + f_2(x). \tag{7b-2}$$

Substituting equations (7*a*-2) and (7*b*-2) into (4*c*), then by employing (7*d*)–(7*g*) and (12*d*)–(12*g*), one finds

$$-\frac{\mathrm{d}f_1(y)}{\mathrm{d}y} = \frac{\mathrm{d}f_2(x)}{\mathrm{d}x}.$$

Omitting trial functions $f_1(y)$, $f_2(x)$, which only give rigid body displacements, one obtains

$$u_x + \mathrm{i}u_y = 32(4c_1c_2 - c_3 - c_1c_4)g_4''(z) - 32(c_1c_4 - c_3)(\overline{g_3'''(z)} + z\overline{g_4'''(z)}) \tag{14a}$$

where

$$c_3 = \frac{R_1^2+R_2^2}{c}, \qquad c_4 = \frac{K_1+K_2}{c}. \tag{14b}$$

Similarly, the complex representations of displacement components of phason fields can be expressed as follows:

$$w_x + \mathrm{i}w_y = \frac{32(R_1 - \mathrm{i}R_2)}{K_1 - K_2}\overline{\Theta(z)}. \tag{15}$$

4. Elliptic notch problem

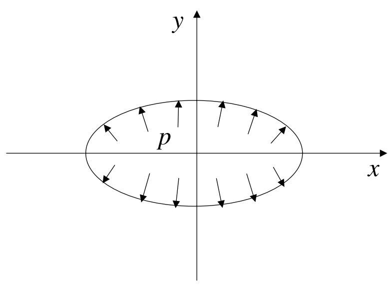

Figure 1. An elliptic notch in a decagonal quasicrystal.

To illustrate the effect of the stress potential and complex variable function method on the complicated stress boundary value problems of higher order partial differential equations originating from quasicrystal elasticity, we here calculate the stress and displacement field induced by an elliptic notch $L(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1)$ (see figure 1), the edge of which is subjected to a uniform pressure p.

The boundary conditions of this problem can be expressed as follows:

$$\sigma_{xx}l + \sigma_{xy}m = X_n, \qquad \sigma_{yy}m + \sigma_{xy}l = Y_n \qquad (x, y) \in L \tag{16}$$

$$H_{xx}l + H_{xy}m = X^h, \qquad H_{yy}m + H_{yx}l = Y^h \qquad (x, y) \in L \tag{17}$$

where $l = \frac{\mathrm{d}y}{\mathrm{d}s}$, $m = -\frac{\mathrm{d}x}{\mathrm{d}s}$, $X_n = -p\cos(n, x)$, $Y_n = -p\cos(n, y)$ denote the components of surface traction, p is the magnitude of the pressure, X^h and Y^h are generalized surface tractions and n represents the outer unit normal vector of any point of the boundary. As the measurement of generalized tractions has not been reported so far for simplicity we assume that $X^h = 0$, $Y^h = 0$.

Similar to Muskhelishvili [19], by equations (9), (11) and (16), one has

$$g_4''(z) + \overline{g_3'''(z)} + z\overline{g_4'''(z)} = \frac{\mathrm{i}}{32c_1}\int (X_n + \mathrm{i}Y_n)\,\mathrm{d}s = -\frac{1}{32c_1}pz \qquad z \in L. \tag{18}$$

Taking the conjugate on both sides of equation (18) yields

$$\overline{g_4''(z)} + g_3'''(z) + \bar{z}g_4'''(z) = -\frac{1}{32c_1}p\bar{z} \qquad z \in L. \tag{19}$$

From equations (9), (11) and (17) we have

$$\begin{aligned} &R_1 \operatorname{Im}\Theta(z) + R_2 \operatorname{Re}\Theta(z) = 0 \qquad z \in L \\ &-R_1 \operatorname{Re}\Theta(z) + R_2 \operatorname{Im}\Theta(z) = 0 \qquad z \in L. \end{aligned} \tag{20}$$

Multiplying the second formula of (20) by i and adding it to the first, one obtains

$$\Theta(z) = 0 \qquad z \in L. \tag{21}$$

Because the function $g_1(z)$ does not appear in the displacement and stress formulae, boundary equations (18), (19) and (21) are sufficient for determining the unknown functions $g_2(z)$, $g_3(z)$ and $g_4(z)$. However, the calculation cannot be completed at the z-plane due to the complexity of the evaluation; we must use the conformal mapping

$$z = \omega(\zeta) = R_0\left(\frac{1}{\zeta} + m\zeta\right) \tag{22}$$

to transform the exterior of the ellipse at the z-plane onto the interior of the unit circle at the ζ-plane (refer to figure 2), where $\zeta = \xi + \mathrm{i}\eta = \rho\mathrm{e}^{\mathrm{i}\varphi}$ and $R_0 = \frac{a+b}{2}$, $m = \frac{a-b}{a+b}$.

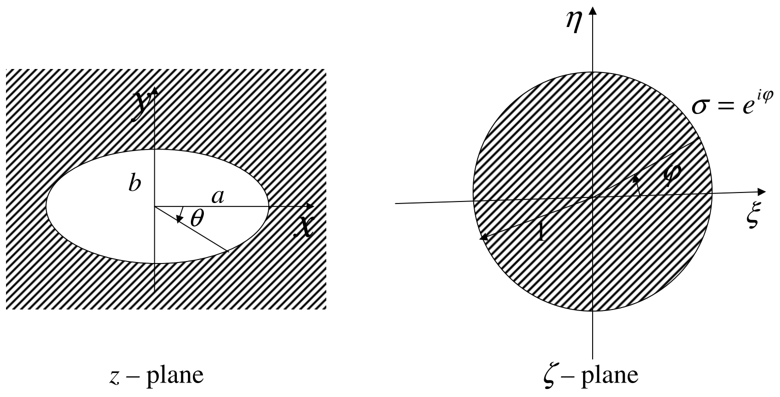

Figure 2. Conformal mapping from the exterior of the elliptic hole at the z-plane onto the interior of the unit circle at the ζ-plane.

For simplicity, we introduce the following new symbols

$$g_2^{(\mathrm{IV})}(z) = h_2(z), \qquad g_3'''(z) = h_3(z), \qquad g_4''(z) = h_4(z).$$

And we have

$$h_i(z) = h_i(\omega(\zeta)) = \Phi_i(\zeta), \qquad h_i'(z) = \frac{\Phi_i'(\zeta)}{\omega'(\zeta)} \qquad (i = 1, 2, 3, 4). \tag{23}$$

Substituting (23) into (18), (19) and (21), then multiplying both sides of the equations by $\frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}\sigma}{\sigma - \zeta}$, and integrating around the unit circle, we have

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\Phi_4(\sigma)\,\mathrm{d}\sigma}{\sigma-\zeta} + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\overline{\Phi_3(\sigma)}\,\mathrm{d}\sigma}{\sigma-\zeta} + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\omega(\sigma)}{\overline{\overline{\omega(\sigma)}}} \frac{\overline{\Phi_4(\sigma)}\,\mathrm{d}\sigma}{\sigma-\zeta} \\
&\qquad = -\frac{P}{32c_1} \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\omega(\sigma)\,\mathrm{d}\sigma}{\sigma-\zeta} \\
&\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\overline{\Phi_4(\sigma)}\,\mathrm{d}\sigma}{\sigma-\zeta} + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\Phi_3(\sigma)\,\mathrm{d}\sigma}{\sigma-\zeta} + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\overline{\omega(\sigma)}}{\omega(\sigma)} \frac{\Phi_4(\sigma)\,\mathrm{d}\sigma}{\sigma-\zeta} \\
&\qquad = -\frac{P}{32c_1} \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\overline{\omega(\sigma)}\,\mathrm{d}\sigma}{\sigma-\zeta} \\
&\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\Phi_2(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \frac{\Phi_3'(\sigma)\,\mathrm{d}\sigma}{\sigma-\zeta} \\
&\qquad + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\overline{\omega(\sigma)}^2 \omega''(\sigma)}{\omega'(\sigma)^3} \frac{\Phi_4'(\sigma)}{\sigma-\zeta}\,\mathrm{d}\sigma = 0
\end{aligned} \tag{24}$$

where $\sigma = \mathrm{e}^{\mathrm{i}\varphi} (\rho = 1)$ represents the value of ζ at the unit circle.

According to the Cauchy integral formula and analytic extension of complex variable function theory, one can obtain the solution of equations (24)

$$\begin{aligned}
\Phi_2(\zeta) &= \frac{pR_0}{32c_1} \frac{\zeta(\zeta^2+m)[(1+m^2)(1+m\zeta^2) - (\zeta^2+m)]}{(m\zeta^2-1)^3} \\
\Phi_3(\zeta) &= \frac{pR_0}{32c_1} \frac{(1+m^2)\zeta}{m\zeta^2-1} \\
\Phi_4(\zeta) &= -\frac{pR_0}{32c_1} m\zeta.
\end{aligned} \tag{25}$$

Utilizing the above mentioned results, the phonon and phason stresses can be determined at the ζ-plane. We here only give a simple example, i.e. along the edge of notch ($\rho = 1$); there are the phonon stress components such as

$$\sigma_{\varphi\varphi} = p\frac{1 - 3m^2 + 2m\cos 2\varphi}{1 + m^2 - 2m\cos 2\varphi}, \qquad \sigma_{\rho\rho} = -p, \qquad \sigma_{\rho\varphi} = \sigma_{\varphi\rho} = 0$$

which are identical to the well-known results of the classical elasticity theory.

5. Elastic field caused by a Griffith crack

The solution of the Griffith crack subjected to a uniform pressure can be obtained corresponding to the case $m = 1$, $R_0 = \frac{a}{2}$ of the above solution. For explicitness, we express the solution in the z-plane. The inversion of transformation (22) is

$$\zeta = \frac{1}{a}(z - \sqrt{z^2 - a^2}). \tag{26}$$

From equations (23), (25) and (26), we have

$$
\begin{aligned}
g_2^{(\mathrm{IV})}(z) &= -\frac{pa^2}{128c_1}\frac{z^2}{\sqrt{(z^2-a^2)^3}} \\
g_3'''(z) &= -\frac{p}{64c_1}\frac{a^2}{\sqrt{z^2-a^2}}, \\
g_4''(z) &= \frac{p}{64c_1}\left(\sqrt{z^2-a^2}-z\right).
\end{aligned} \tag{27}
$$

So the stresses and the displacements can be expressed with a complex variable z (see appendix A for details).

Similar to classical elasticity theory, we introduce three pairs of polar coordinates (r,θ), (r_1,θ_1) and (r_2,θ_2) with their origins at the crack centre, at the right crack tip and at the left tip, i.e. $z = r\mathrm{e}^{\mathrm{i}\theta}$, $z-a = r_1\mathrm{e}^{\mathrm{i}\theta_1}$, $z+a = r_2\mathrm{e}^{\mathrm{i}\theta_2}$, respectively. The analytical expressions for the stress and displacement fields can then be obtained (see appendix B for details).

Moreover, the stress intensity factor and free energy of the crack can be evaluated as the direct results of the solution. We here only list the stress intensity factor and energy release rate as below

$$
K_I = \sqrt{\pi a}\,p, \qquad \mathcal{G} = \frac{(L+2M)(K_1+K_2)+2(R_1^2+R_2^2)}{8(L+M)c}(K_I)^2. \tag{28}
$$

It is evident that the present solution covers the solution for point group $10mm$ quasicrystals, or we can say that the solution of the latter is a special case of the present work.

6. Discussion and conclusion

Defects play a central role in the study of the mechanical behaviour of quasicrystals, which is a very difficult problem due to their complicated configuration. It is well known that the Green function method, the Fourier method and other methods cannot solve the notch problem. In addition, the displacement potential function formulation is relatively complicated and the notch problem is very hard to solve by this formulation even if one uses the complex variable function method. To solve the problem it is necessary to develop an effective theory and method. From this work, we can conclude that the stress potential and complex variable function method are powerful for the complicated stress boundary value problem of quasicrystals. Although the general solution (11) is expressed by four analytic functions, we need only three of them to express the stress and displacement components of phason fields, and two of them are necessary for phonon fields. This greatly simplifies the solution procedure for more complicated problems as well.

The main purposes of the present study are to reveal the effect of the notch and the crack on quasicrystals and to compare the current results with those for conventional materials. All field variables for the notch problem are exactly determined in this paper. The solution of the Griffith crack problem as a special case of the results is also given in appendix B. This indicates that the distribution of the phonon stress field is identical to the corresponding results in linear elasticity fracture mechanics (LEFM), while the phason stress field, which arises from the coupling relationship between the phonon and the phason fields, is particular for quasicrystals and also exhibits the square root singularity around the crack tip. The displacement field and the energy release rate in quasicrystals are different from the well known results in LEFM. After a little direct calculation, we can see that these quantities are respectively the sum of two parts: one part in agreement with the results of their counterparts in LEFM, and the other produced by the phonon–phason coupling relationship in quasicrystals. In the case of the critical state, since $\mathcal{G}_{\text{critical}} = 2\gamma$ for brittle materials (for example see [20]), with γ denoting the unit surface energy, from equation (28) we know that, owing to the contribution of phonon–phason coupling, the unit surface energy γ for quasicrystals with point 10 is greater than that for conventional crystals with the same phonon elastic constants and fracture toughness. The above results show that the conclusion in LEFM cannot be directly applied to quasicrystal linear elasticity fracture mechanics (QLEFM). Of course, if the coupling elastic constants R_1, R_2 are so small that they can be ignored, the present results reduce to the well known results in LEFM. Formulae in appendix B are exact solutions for any point, from which one can obtain the asymptotic solutions for a crack tip $r_1/a \ll 1$. When $R_2 = 0$, $R_1 = R$, the material is reduced to the point $10mm$ quasicrystals, and the results are reduced to those for point group $10mm$.

Acknowledgment

This work is supported by the National Natural Science Foundation of China under grant no 10372016 and the Natural Science Foundation of Inner Mongolia under grant no 200607010104.

Appendix A. The details of complex representation of the crack solution

From (12) and (27), the stress fields can be obtained as follows

$$\begin{aligned}\sigma_{xx} &= -32c_1\,\mathrm{Re}(\Omega(z) - 2g_4'''(z))\\ &= -32c_1\,\mathrm{Re}\left[\frac{pa^2(z-\bar{z})}{64c_1\sqrt{(z^2-a^2)^3}} - \frac{p}{32c_1}\left(\frac{z}{\sqrt{z^2-a^2}} - 1\right)\right]\\ &= p\,\mathrm{Re}\left(\frac{z}{\sqrt{z^2-a^2}} - \frac{\mathrm{i}a^2y}{\sqrt{(z^2-a^2)^3}} - 1\right)\end{aligned} \tag{A.1}$$

$$\begin{aligned}\sigma_{yy} &= 32c_1\,\mathrm{Re}(\Omega(z) + 2g_4'''(z))\\ &= 32c_1\,\mathrm{Re}\left[\frac{pa^2(z-\bar{z})}{64c_1\sqrt{(z^2-a^2)^3}} + \frac{p}{32c_1}\left(\frac{z}{\sqrt{z^2-a^2}} - 1\right)\right]\\ &= p\,\mathrm{Re}\left(\frac{z}{\sqrt{z^2-a^2}} + \frac{\mathrm{i}a^2y}{\sqrt{(z^2-a^2)^3}} - 1\right)\end{aligned} \tag{A.2}$$

$$\begin{aligned}\sigma_{xy} &= \sigma_{yx} = 32c_1\,\mathrm{Im}\,\Omega(z)\\ &= p\,\mathrm{Im}\,\frac{\mathrm{i}a^2y}{\sqrt{(z^2-a^2)^3}}\end{aligned} \tag{A.3}$$

$$\begin{aligned}H_{xx} &= 32R_1\,\mathrm{Re}(\Theta'(z) - \Omega(z)) - 32R_2\,\mathrm{Im}(\Theta'(z) - \Omega(z))\\ &= pa^2\left[R_1\,\mathrm{Re}\left(\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}} - \frac{z-\bar{z}}{c_1\sqrt{(z^2-a^2)^3}}\right)\right.\\ &\quad\left. - R_2\,\mathrm{Im}\left(\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}} - \frac{z-\bar{z}}{c_1\sqrt{(z^2-a^2)^3}}\right)\right]\end{aligned} \tag{A.4}$$

$$\begin{aligned}H_{xy} &= -32R_1\,\mathrm{Im}(\Theta'(z) + \Omega(z)) - 32R_2\,\mathrm{Re}(\Theta'(z) + \Omega(z))\\ &= -pa^2\left(R_1\,\mathrm{Im}\,\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}} + R_2\,\mathrm{Re}\,\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}}\right)\end{aligned} \tag{A.5}$$

$$H_{yx} = -32R_1\,\mathrm{Im}\left(\Theta'(z) - \Omega(z)\right) - 32R_2\,\mathrm{Re}(\Theta'(z) - \Omega(z))$$

$$= -pa^2\left[R_1\,\mathrm{Im}\left(\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}} - \frac{z-\bar{z}}{c_1\sqrt{(z^2-a^2)^3}}\right)\right.$$

$$\left.+ R_2\,\mathrm{Re}\left(\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}} - \frac{z-\bar{z}}{c_1\sqrt{(z^2-a^2)^3}}\right)\right] \tag{A.6}$$

$$H_{yy} = -32R_1\,\mathrm{Re}(\Theta'(z) + \Omega(z)) + 32R_2\,\mathrm{Im}(\Theta'(z) + \Omega(z))$$

$$= pa^2\left(-R_1\,\mathrm{Re}\,\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}} + R_2\,\mathrm{Im}\,\frac{3z(z-\bar{z})^2}{4c_1\sqrt{(z^2-a^2)^5}}\right). \tag{A.7}$$

By equations (14), (15) and (27), we have

$$u_x + \mathrm{i}u_y = \frac{p}{2}\left(\frac{1}{L+M} - \frac{c_3}{c_1}\right)(\sqrt{z^2-a^2} - z)$$

$$- \frac{p}{2}\left(c_4 - \frac{c_3}{c_1}\right)\overline{\left(\bar{z}\left(\frac{z}{\sqrt{z^2-a^2}} - 1\right) - \frac{a^2}{\sqrt{z^2-a^2}}\right)} \tag{A.8}$$

$$w_x + \mathrm{i}w_y = -\frac{pa^2(R_1 - \mathrm{i}R_2)}{4c_1(K_1 - K_2)}\overline{\frac{(z-\bar{z})^2}{\sqrt{(z^2-a^2)^3}}}. \tag{A.9}$$

Appendix B. The details of real representation of the crack solution

From (A.1)–(A.9), the stress fields can be expressed as

$$\sigma_{xx} = p[r(r_1r_2)^{-\frac{1}{2}}\cos(\theta - \bar{\theta}) - a^2r(r_1r_2)^{-\frac{3}{2}}\sin\theta\sin\tfrac{3}{2}\bar{\theta} - 1] \tag{B.1}$$

$$\sigma_{yy} = p[r(r_1r_2)^{-\frac{1}{2}}\cos(\theta - \bar{\theta}) + a^2r(r_1r_2)^{-\frac{3}{2}}\sin\theta\sin\tfrac{3}{2}\bar{\theta} - 1] \tag{B.2}$$

$$\sigma_{xy} = \sigma_{yx} = pa^2r(r_1r_2)^{-\frac{3}{2}}\sin\theta\cos 3\bar{\theta} \tag{B.3}$$

$$H_{xx} = -\frac{R_1pa^2}{c_1}(3r^3(r_1r_2)^{-\frac{5}{2}}\sin^2\theta\cos(\theta - 5\bar{\theta}) + 2r(r_1r_2)^{\frac{3}{2}}\sin\theta\sin 3\bar{\theta})$$

$$+ \frac{R_2pa^2}{c_1}(3r^3(r_1r_2)^{-\frac{5}{2}}\sin^2\theta\sin(\theta - 5\bar{\theta}) + 2r(r_1r_2)^{-\frac{3}{2}}\sin\theta\cos 3\bar{\theta}) \tag{B.4}$$

$$H_{xy} = \frac{3pa^2}{c_1}r^3(r_1r_2)^{-\frac{5}{2}}\sin^2\theta[R_1\sin(\theta - 5\bar{\theta}) + R_2\cos(\theta - 5\bar{\theta})] \tag{B.5}$$

$$H_{yx} = \frac{R_1pa^2}{c_1}(3r^3(r_1r_2)^{-\frac{5}{2}}\sin^2\theta\sin(\theta - 5\bar{\theta}) + 2r(r_1r_2)^{-\frac{3}{2}}\sin\theta\cos 3\bar{\theta})$$

$$+ \frac{R_2pa^2}{c_1}(3r^3(r_1r_2)^{-\frac{5}{2}}\sin^2\theta\cos(\theta - 5\bar{\theta}) + 2r(r_1r_2)^{-\frac{3}{2}}\sin\theta\sin 3\bar{\theta}) \tag{B.6}$$

$$H_{yy} = \frac{3pa^2}{c_1}r^3(r_1r_2)^{-\frac{5}{2}}\sin^2\theta[R_1\cos(\theta - 5\bar{\theta}) - R_2\sin(\theta - 5\bar{\theta})] \tag{B.7}$$

in which $\bar{\theta} = \frac{\theta_1+\theta_2}{2}$.

The displacement fields can be calculated using equations (A.8) and (A.9) as

$$u_x = \frac{p}{2}\left(\frac{1}{L+M} - \frac{c_3}{c_1}\right)(r_1r_2)^{\frac{1}{2}}\cos\bar{\theta} - \frac{p}{2}\left(\frac{1}{L+M} - c_4\right)r\cos\theta$$

$$- \frac{p}{2}\left(c_4 - \frac{c_3}{c_1}\right)(r^2 - a^2)(r_1r_2)^{-\frac{1}{2}}\cos\bar{\theta} \tag{B.8}$$

$$u_y = \frac{p}{2}\left(\frac{1}{L+M} - \frac{c_3}{c_1}\right)(r_1 r_2)^{\frac{1}{2}}\sin\overline{\theta} - \frac{p}{2}\left(\frac{1}{L+M} - c_4\right) r\sin\theta - \frac{p}{2}\left(c_4 - \frac{c_3}{c_1}\right)(r^2 - a^2)(r_1 r_2)^{-\frac{1}{2}}\sin\overline{\theta} \tag{B.9}$$

$$w_x = \frac{pa^2}{c_1} r^2 (r_1 r_2)^{-\frac{3}{2}} \sin^2\theta (R_1 \cos 3\overline{\theta} + R_2 \sin 3\overline{\theta}) \tag{B.10}$$

$$w_y = \frac{pa^2}{c_1} r^2 (r_1 r_2)^{-\frac{3}{2}} \sin^2\theta (R_1 \sin 3\overline{\theta} - R_2 \cos 3\overline{\theta}). \tag{B.11}$$

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